

Stationary Phase and the Theory of Measurement

— $1/N$ expansion —

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The measuring process is studied, where a macroscopic number N of particles in the detector interact with the object. When $N \rightarrow \infty$, the fluctuation of the object between different eigen-states of the operator O to be measured is suppressed, frozen to one and the same state while the detector is on. During this period, the stationary phase accompanying the macrovariable is established to have a one to one correspondence with the eigen-value of O . A model is studied which produces the ideal result when $N \rightarrow \infty$ and the correction terms are calculated in powers of $1/N$. It is identical to the expansion including the fluctuation of the object successively.

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I. INTRODUCTION

The observational problem in quantum mechanics has a long history of debates [1, 2, 3, 4, 5, 6, 7, 8]. In particular, the role of the decoherence due to environmental influence has been widely discussed [9, 10, 11, 12, 13]. The dynamical reduction model has actually been constructed and many recent researches are focused on this subject [14, 15, 16, 17, 18]. Irrespectively of the mechanism that leads to the reduction, we need the statistical treatment of the data, which is done by using the wave function following the rule of the ordinary quantum mechanics. When we apply the quantum mechanics to the detector system, composed of a number of microscopic particles, two requirements have to be fulfilled; (1) any detector variable $X(t)$, the pointer position for example, should show a non-fluctuating behavior of classical type as a function of the time. (2) different eigen-states of object operator O to be measured have to be mapped onto different values of $X(t)$.

We stress here that both (1) and (2) are realized by the stationary phase accompanying any macroscopic variables [19] (The precise definition of the macrovariable is given in Sec.II A). Consider (1) in the path-integral form [20]. Out of many fluctuating paths, the stationary phase selects one particular smooth path denoted as $X^{\text{st}}(t)$. Then the absolute square of the wave function of X integrated by other degrees has a non-diffusive peak, equal to the density of a classical point-like particle $\delta(X - X^{\text{st}}(t))$. This is what we observe as a macroscopic object in the ordinary life. As for (2), we recall that the measuring device is prepared in such a way that O interacts, directly or indirectly, with a large number N of particles in the detector. So the Hamiltonian H_I describing such an interaction may be a function of O , coordinates x_i 's and the momenta p_i 's of all these particles and is $O(N)$. Since the Hamiltonian of the object H_O itself is $O(1)$, it can be

neglected compared with H_I . Then, for $N \rightarrow \infty$, *each eigen-state of O is frozen in the same state as long as the detector is on. Moreover, precisely during this period, the detector variable $X(t)$ changes its value depending on the eigen-values of O .*

The desired mapping is realized in this way. Note that if the object interacts with a finite number of particles, the process is simply a quantum mechanical one, nothing to do with the measurement.

The above observation suggests the $1/N$ expansion scheme for large N , which incorporates successively the fluctuation of the object connecting different eigen-states of O . It is the purpose of this paper to show that this is indeed the case by adopting a simple model of separable X . It is solvable by $1/N$ expansion and we calculate several lower order terms explicitly. The stationary phase was applied to the macrovariable in [19] in the lowest order but above observations were lacking.

In Sec.II, crucial points of the stationary phase are presented on which subsequent discussions are based. These are not stated in [19]. Following Sec.II, a model is defined in Sec.III, and the signal function is introduced, which agrees with the density of the classical point-like particle for $N \rightarrow \infty$. Higher order corrections are studied in Sec.IV. They are given by the expansion in (fractional) powers of $1/N$ and the result is summarized in (85) below. This type of calculation is surely required since in the actual detector N is finite, although very large, and the correction terms might be tested experimentally. An attempt at the numerical estimate of the correction terms is presented. After the discussions in Sec.V, general non-separable case is discussed in Appendix A. In Appendixes B,C,D, some of calculational details are shown and in E, the order estimation of neglected terms is given.

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II. STATIONARY PHASE AND MACROVARIABLE

A. Macroscopic system and macrovariable

Before constructing the model of measurement, the properties of macro and microvariables have to be elucidated, since the quantum mechanical detection process consists of an interplay between the two. Consider a macroscopic system, which contains a large number of microscopic particles with coordinates x_i ($i = 1, 2, \dots, N$), all of which are assumed to have the same mass m . To make formulas simple, we work below in one dimension, extension to three dimensional case being straightforward. Now an extensive quantity grows up with the size of the system. As has been introduced in [19] in the case of field theory, a macrovariable of N particle system is an intensive quantity defined by dividing an extensive quantity by N . The center of mass, $X = \sum_{i=1}^N x_i/N$ is the simplest example.

To make above definition more precise in quantum case, let us recall that the quantum mechanical process is described by the path-integral form given by

$$\int \prod_{i=1}^N [dx_i] e^{iS/\hbar} = \int \prod_{i=1}^N [dx_i] e^{i \int dt L/\hbar}, \quad (1)$$

where $S = \int dt L$ is the action functional of the system. The Lagrangian $L = L([x_i], [\dot{x}_i])$ is an extensive quantity, so it is $O(N)$. ($[x_i]$ represents x_i 's collectively.) For this statement to hold, the system has to be thermodynamically normal, which holds when the interaction among microdegrees is of short range and the particle density is finite over all space. (The statistical factor $1/N!$ has to be inserted properly for the system of identical particles.)

The macroscopic system behaves classically as a whole, while the system at the same time contains atoms and molecules, which is described quantum mechanically. The stationary phase accompanying the macrovariable just realizes such a picture. The reason is simple; if we change one x_i as $x_i \rightarrow x_i + a$ by a finite amount a , then the change of the phase $S[x_i]$ is of the order a . On the other hand, suppose the macrovariable X changes by a finite amount a . It means that macroscopic number N of x_i 's are displaced by the order of a . Then $S[x_i]$ in general shifts by $O(Na)$. Now we integrate over all values x_i 's in (1), so when $N \rightarrow \infty$ only that point of $S[x_i]$ contributes which is stationary when X varies. Since this holds for any time slice of the path-integral (1), one gets a smooth stationary trajectory of X . It describes the motion of a macroscopic body as a whole. On the other hand, no stationary point exists for each x_i , so every x_i remains as a fluctuating quantum variable. Consider instead the limit $\hbar \rightarrow 0$. The change in $S[x_i]/\hbar$ under $x_i \rightarrow x_i + a$ is $O(a/\hbar)$ for each i , so every x_i is determined by the stationary equation, i.e. the Newtonian equation. It is not the "classical limit" as seen in the ordinary life.

B. Particle picture by the stationary phase

The above statement is seen clearly if we take the separable case of the center of mass for X . (See Appendix A for non-separable case.) We write $x_i = X + x'_i$, where x'_i is the coordinate measured from the center of mass and satisfies $\sum_i x'_i = 0$. Now (1) becomes

$$\int [dX] e^{iS_1[X]/\hbar} \int \prod_{i=1}^N [dx'_i] \delta(\sum_i x'_i) e^{iS_2[x'_i]/\hbar}. \quad (2)$$

The second factor describes the microscopic quantum phenomena and can be neglected, or integrated out, in the measurement theory, since we are interested only in X . The first factor accompanies the stationary phase since, as discussed above, $S_1[X]$ is proportional to N ; $S_1[X] = N s_1[X]$. For large N , the X integration is dominated by the solution of functional stationary equation $\delta s_1[X]/\delta X(t) = 0$. The fluctuation of X is suppressed by the phase cancellation and a single smooth trajectory $X^{\text{st}}(t)$ is selected by the constructive phase coherence among paths near $X^{\text{st}}(t)$. Although the fluctuation of X defines the wave function $\Psi(X, t)$ for finite N , once N becomes infinity, X reduces to the variable of a point-like particle. These statements are based on the following well-known formula. Let $f(X)$ be a function of X having the stationary point at X_0 , then we have

$$\lim_{N \rightarrow \infty} e^{iNf(X)} = e^{iNf(X_0)} \sqrt{\frac{2\pi i}{Nf''(X_0)}} \delta(X - X_0). \quad (3)$$

Consider here the Feynman kernel $K(X, T; Y, 0)$, which connects the wave functions at different times;

$$\Psi(X, T) = \int K(X, T; Y, 0) \Psi(Y, 0) dY. \quad (4)$$

Applying (3) at every time slice from $t = 0$ to $t = T$, the kernel is seen to contain a factor $\delta(X - X^{\text{st}}(Y, T))$, where $X^{\text{st}}(Y, T)$ passes Y at $T = 0$. (Initial velocity depends on the form of $\Psi(X, 0)$.) Thus, each point X on the wave function $\Psi(X, 0)$ is just transported along $X^{\text{st}}(Y, T)$, as opposed to the Huygens picture of wave mechanics. Our $\Psi(X, T)$ here represents the mixed state. If we choose $|\Psi(Y, 0)|^2 = \delta(Y - X_I)$, then $|\Psi(X, T)|^2 = \delta(X - X^{\text{st}}(X_I, T))$; the wave function has a non-diffusive sharp peak, representing the density of a point-like particle. Explicit examples appear later.

To discuss other local densities, let us discretize the time with the interval Δt . Then the fluctuating momentum operator [20] $P = (\hbar/i)\partial/\partial X(t) = Nm(X(t + \Delta t) - X(t))/\Delta t$ becomes the classical expression $Nm\dot{X}(t)$ evaluated along $X^{\text{st}}(t)$ as $N \rightarrow \infty$. The quantum mechanical expression of the momentum or the energy e.t.c. reduces to the corresponding classical density;

$$\begin{aligned} \Psi^*(X, t) (1, P, P^2/2Nm) \Psi(X, t) \longrightarrow \\ (1, Nm\dot{X}(t), Nm\dot{X}^2/2) \delta(X - X^{\text{st}}(t)). \end{aligned}$$

C. Measurement by macrovariable

Suppose we measure the object operator O by the interaction Hamiltonian $H_I(O, [x_i], [p_i])$. Then the total Hamiltonian is the sum of three terms;

$$H = H_O + H_D([x_i], [p_i]) + H_I(O, [x_i], [p_i]), \quad (5)$$

where H_O is the object Hamiltonian, H_D that of the detector. As stated in the Introduction, O interacts with many particles in the detector and X is so chosen that it includes (almost) all of them;

$$H_I = \sum_{i=1}^N h(O, x_i, p_i), \quad X = \sum_{i=1}^N g(x_i, p_i)/N \quad (6)$$

with some functions h and g . Here and hereafter, N is the number of particles involved in (6). H_D in (5) is the Hamiltonian of these particles and is $O(N)$ together with H_I . Arguments of Secs II A, II B can be applied to X thus defined.

One comment here. If one can find any parameter α which produces the stationary phase for some variable η , then α and η can be used in place of N and X . Then, the macrocity will not be required for the measurement.

D. Freezing the object state

Now in general, O does not commute with H_O ,

$$[H_O, O] \neq 0. \quad (7)$$

So, the eigenstates of O always fluctuates among different states. To measure such a fluctuating object, we have to suppress the size of the fluctuation somehow. This is done by taking $H_I = O(N)$; since $H_O = O(1)$, we can neglect H_O for $N \rightarrow \infty$ and adopt $H = H_D + H_I$ as the Hamiltonian. Thus, taking the representation which diagonalizes O , $O|a\rangle = \lambda_a|a\rangle$, each eigen-state $|a\rangle$ develops by $H_D + H_I(\lambda_a, [x_i], [p_i]) \equiv H_D + H_{I,a}$ and remains in the same state $|a\rangle$ as long as $H_I \neq 0$, i.e. the detector is on. Precisely during this period of $H_I \neq 0$, the macrovariable can change its value since it evolves by $H_D + H_{I,a}$, which produces different stationary phases for different λ_a 's since H_I is $O(N)$. Thus we get different values of X depending on the microscopic state of the object. It is realized independently of the detailed forms of the functions h or g above, as long as both include large number of particles which interact with O . This is the amplification mechanism of the detection process in terms of the stationary phase. Higher order correction terms for large N are given systematically through the expansion in powers of the off-diagonal elements of the object Hamiltonian H_O . Below, this is done explicitly based on the model Hamiltonian.

III. THE MODEL

A. The Hamiltonian

When we construct the model, above functions $h(O, x_i, p_i)$ and $g(x_i, p_i)$ has to be fixed. The simplest case is $h = -fOx_i$ and $g = x_i$ (f is the coupling strength.); then

$$H_I = -\sum_{i=1}^N fOx_i = -fNOX. \quad (8)$$

Here X is the center of mass of particles that interact with O . In the realistic detector, the photomultiplier for example, complicated processes may happen. An object interacts with an atom in the detector (via exchanging a photon), ionizing an electron. It is accelerated by the electric field applied in certain direction, which interacts with another atom, ionizing a second electron and so force, until we have a macroscopic number of electrons, giving a signal as the current. Or a high energy object interacts with many atoms in the direction on its momentum, along which the track of ionized electrons is detected. Above H_I simulates these processes by a direct interaction of O and N particles, which are in the direction of applied electric field or in the direction of the object momentum. So the problem can be simulated by one dimension, with X taken to be the center of mass of x_i in that direction. Independently of the detailed form of the interaction, the essential point is that, although each electron receives a microscopic amount of energy, the sum of them is $O(N)$, which affects the stationary phase of X . Since H_D is $P^2/2Nm = Nm\dot{X}^2/2$ ($P = Nm\dot{X}$ is the total momentum) plus terms independent of X , the total Hamiltonian, the object plus detector, of our model is defined by

$$H = H_O + Nm\dot{X}^2/2 - NfOX \equiv H_O + H_N, \quad (9)$$

$$H_N = Nm\dot{X}^2/2 - NfOX. \quad (10)$$

We do not write x'_i part hereafter, since the dependence on X and x'_i is factorized as (2).

B. The initial wave function

Below the eigen-values λ_a of O are assumed to be discrete and non-degenerate; $\lambda_a \neq \lambda_b$ if $a \neq b$. Writing the eigenstate of X as $|X\rangle$ with continuous eigen-value X , the complete set of states of our model Hamiltonian is given by $|a\rangle|X\rangle \equiv |a, X\rangle$. We also use the complete set spanned by $|x, X\rangle$. The eigen-function is then $\phi_a(x) = \langle x|a\rangle$.

Let the detector be switched on at $t = 0$, and the state vector at $t = 0$ is written as $|\Psi\rangle = |\phi\rangle|\Phi\rangle$, which is the product of the object $|\phi\rangle$ and of the detector $|\Phi\rangle$. Expanding as $|\phi\rangle = \sum_a C_a|a\rangle$ into complete sets, the

initial wave function is given by

$$\begin{aligned} \Psi(x, X, t=0) &= \langle x, X | \Psi \rangle = \langle x | \phi \rangle \langle X | \Phi \rangle, \\ \langle x | \phi \rangle &= \sum_a C_a \langle x | a \rangle = \sum_a C_a \phi_a(x), \end{aligned} \quad (11)$$

The initial wave function of the detector $\Psi(X)$ is assumed to have a peak at some value of X , with the precision Δ . To be explicit, we adopt a Gaussian type;

$$\langle X | \Phi \rangle = (\pi\Delta^2)^{-1/4} \exp(-X^2/2\Delta^2). \quad (12)$$

$|\Psi(X)|^2$ becomes $\delta(X)$ when $\Delta \rightarrow 0$. The center position is at $X = 0$ and the initial velocity $\dot{X} = (\hbar/Nmi)\partial/\partial X$ is also zero when $N \rightarrow \infty$. (Non-zero velocity v is obtained by multiplying $\exp(iNmvX/\hbar)$ to (12).)

When $N \rightarrow \infty$, the center of the peak traces a classical trajectory determined by H_N keeping the width Δ constant. Here $\langle X | \Psi \rangle^2$ represents the density matrix of a mixed state, with Δ representing the classical uncertainty. If N is large but finite, X fluctuates and the diffusion process comes in. The numerical consideration is given at the end of Sec.III E, where we will see that, up to the order we are considering, the influence of the diffusion is negligible in the detection process and the dominant effect comes from the fluctuation of the object while the detector is on.

Now the macroscopic limit is $N \rightarrow \infty$, with other quantities kept fixed. But in order to avoid the classical uncertainty and keep various formulas simple, we take the limit $\Delta \rightarrow 0$ of the coefficients of limiting expression obtained by $N \rightarrow \infty$. Such a limit, first $N \rightarrow \infty$, then $\Delta \rightarrow 0$, is denoted as \Rightarrow .

C. Time evolution and the expansion by the off diagonal elements of H_O

The total wave function develops in time as

$$\begin{aligned} \Psi(x, X, T) &= \langle x, X | \exp(-iHT/\hbar) | \Psi \rangle \\ &= \langle x, X | \exp(-iHT/\hbar) \sum_a C_a | a \rangle \langle X | \Phi \rangle. \end{aligned} \quad (13)$$

When we expand in powers of H_O , we first sum up the diagonal term $(H_O)_{aa}$ exactly in every order of expansion. Then the expansion becomes the one in terms of the power of off-diagonal elements $(H_O)_{ab}$, $a \neq b$. To achieve this, we use the well-known formula;

$$\begin{aligned} U(T) &= \exp(-iHT/\hbar) = U_N(T)U_O(T), \\ U_N(T) &= \exp(-iH_N T/\hbar), \\ U_O(T) &= T \exp\left(-i \int_0^T ds H_O(s)/\hbar\right), \end{aligned} \quad (14)$$

$$H_O(s) = U_N^\dagger(s) H_O U_N(s). \quad (15)$$

In (14), T implies the time ordering operation. To get the desired expansion, consider

$$\begin{aligned} H_O(s) | a \rangle &= U_N^\dagger(s) H_O U_N(s) | a \rangle \\ &= U_{N,a}^\dagger(s) H_O | a \rangle U_{N,a}(s) \\ &= U_{N,a}^\dagger(s) (H_O)_{aa} | a \rangle U_{N,a}(s) + \text{off diagonal part} \\ &= (H_O)_{aa} | a \rangle + \text{off diagonal part}. \end{aligned} \quad (16)$$

Here we have introduced

$$U_{N,a}(s) = \exp(-iH_{N,a}s/\hbar), \quad (17)$$

$$H_{N,a} = Nm\dot{X}^2/2 - Nf_a X, \quad f_a = f\lambda_a. \quad (18)$$

Thus, when we sum up the diagonal elements of H_O , H_O can be treated as a c-number and the diagonal parts are summed up into the phase. Thus we can use the formula

$$\begin{aligned} U_N(T)U_O(T) | a \rangle &= U_{N,a}(T) \exp(i\theta_a T) | a \rangle, \\ \theta_a &= -(H_O)_{aa}/\hbar. \end{aligned} \quad (19)$$

Now we concentrate on the defining equation of the T-product (14). It is an infinite product of the term $\exp(-iH_O(s)\Delta s/\hbar)$ in the infinitesimal time interval Δs . When it is evaluated by off-diagonal elements $[H_O]^{\text{nd}}$, we write $\exp(-iH_O(s)\Delta s/\hbar) \sim 1 + (-iH_O(s)\Delta s/\hbar)$. Using eqs.(16) and (19), the expansion thus obtained becomes

$$\langle b, X | U_N(T)U_O(T) | a, Y \rangle \Rightarrow \sum_{k=0}^{\infty} U_{ba}^{(k)}(X, Y), \quad (20)$$

where $U^{(k)}$ involves k -th power of $[H_O]^{\text{nd}}$. The results of first few terms are shown below; ($[U_O(T)]^{\text{d}}$ contains the diagonal parts only, while $[H_O]^{\text{nd}}$ off-diagonal parts.)

$$\begin{aligned} U_{ba}^{(0)}(X, Y) &= \langle b, X | U_N(T) | a, Y \rangle \\ &= \delta_{ab} \exp(i\theta_a T) \langle X | U_{N,a}(T) | Y \rangle, \\ U_{ba}^{(1)}(X, Y) &= \langle b, X | U_N(T) (-i/\hbar) \\ &\quad \times \int_0^T ds [U_O(T-s)]^{\text{d}} [H_O(s)]^{\text{nd}} [U_O(s)]^{\text{d}} | a, Y \rangle \\ &= (-i/\hbar) (H_O)_{ba} \int_0^T ds \exp\{i(\theta_b(T-s) + \theta_a s)\} \\ &\quad \times \langle X | U_{N,b}(T-s) U_{N,a}(s) | Y \rangle. \end{aligned} \quad (21)$$

In (22) and in what follows, $([H_O]^{\text{nd}})_{ba}$ is written simply as $(H_O)_{ba}$ for the notational simplicity so $a \neq b$ is implied. The factor δ_{ab} in (21) implies that the object does not fluctuate in the lowest order while the detector is on. By a similar manipulation,

$$\begin{aligned} U_{ba}^{(2)}(X, Y) &= (-i/\hbar)^2 \sum_c (H_O)_{bc} (H_O)_{ca} \\ &\quad \times \int_0^T ds' \int_0^{s'} ds \exp\{i(\theta_b(T-s') + \theta_c(s'-s)) + \theta_a s\} \\ &\quad \times \langle X | U_{N,b}(T-s') U_{N,c}(s'-s) U_{N,a}(s) | Y \rangle. \end{aligned} \quad (22)$$

The wave function has the corresponding expansion

$$\begin{aligned}\Psi(x, X, T) &= \langle x, X | U_N(T) U_O(T) \sum_a C_a | a, \Phi \rangle \\ &= \int dY \sum_{b,a} \langle x | b \rangle C_a \sum_{k=0}^{\infty} U_{ba}^{(k)}(X, Y) \langle Y | \Phi \rangle \\ &\equiv \sum_b \langle x | b \rangle \sum_{k=0}^{\infty} \Psi^{(k)}(b, X, T) = \sum_{k=0}^{\infty} \Psi^{(k)}(x, X, T).\end{aligned}$$

The wave function in the $|a\rangle$ representation

$$\Psi(b, X, T) = \langle b, X | U(T) | \Psi \rangle = \sum_{k=0}^{\infty} \Psi^{(k)}(b, X, T) \quad (24)$$

has been introduced and each $\Psi^{(k)}$ has k -th power of $[H_O]^{\text{nd}}$. In case $[H_O, O] = 0$, $(H_O)_{ba}(\lambda_a - \lambda_b) = 0$ follows, so one gets $[H_O]^{\text{nd}} = 0$. Thus, in (20) or (24) only the lowest term with $k = 0$ is non-vanishing.

D. The signal function

In the real experiment, the object is not actually observed, so let us define the signal function by integrating (summing up) $|\Psi|^2$ by $x(b)$;

$$J(X, T) = \int dx |\Psi(x, X, T)|^2 = \sum_b |\Psi(b, X, T)|^2 \quad (25)$$

$$= \sum_{k=0}^{\infty} J^{(k)}(X, T), \quad (26)$$

$$J^{(0)}(X, T) = \sum_b |\Psi^{(0)}(b, X, T)|^2, \quad (27)$$

$$J^{(1)}(X, T) = \sum_b \Psi^{(0)*}(b, X, T) \Psi^{(1)}(b, X, T) + \text{c.c.}, \quad (28)$$

$$\begin{aligned}J^{(2)}(X, T) &= J_1^{(2)}(X, T) + J_2^{(2)}(X, T), \\ J_1^{(2)}(X, T) &= \sum_b \Psi^{(1)*}(b, X, T) \Psi^{(1)}(b, X, T),\end{aligned} \quad (29)$$

$$J_2^{(2)}(X, T) = \sum_b \Psi^{(0)*}(b, X, T) \Psi^{(2)}(b, X, T) + \text{c.c.} \quad (30)$$

Here $J^{(k)}(X, T)$ is of the order of $([H_O]^{\text{nd}})^k$. Let us calculate $J^{(0,1,2)}(X, T)$ and $\Psi^{(0,1,2)}$ successively.

E. $\Psi^{(0)}(b, X, T)$

The lowest term is calculated by (19), (21) and (24) as

$$\begin{aligned}\Psi^{(0)}(b, X, T) &= \sum_a C_a \langle b, X | U_N(T) | a, \Phi \rangle \\ &= \int dY \exp(i\theta_b T) C_b \langle X | U_{N,b}(T) | Y \rangle \langle Y | \Phi \rangle.\end{aligned}$$

Now we insert (12), and use the following result of the Feynman kernel [20] for the Hamiltonian $H_{N,a}$ of (18);

$$\langle X | U_{N,a}(T) | Y \rangle = \sqrt{\frac{Nm}{2\pi i \hbar T}} e^{iN\Theta_a/\hbar} e^{i\tilde{\theta}_a}, \quad (31)$$

$$\begin{aligned}\Theta_a &= \frac{m(X - Y - \xi_a(T))^2}{2T}, \\ \xi_a(T) &= f_a T^2 / 2m,\end{aligned} \quad (32)$$

$\xi_a(T)$ is the classical change of X during T in the presence of the constant force f_a , with the initial condition $X = \dot{X} = 0$. $\tilde{\theta}_a$ is the classical action for this motion. By (3),

$$\lim_{N \rightarrow \infty} \langle X | U_{N,a}(T) | Y \rangle = \delta(X - Y - \xi_a(T)) e^{i\tilde{\theta}_a}. \quad (33)$$

Apart from the phase, the whole wave function develops as a parallel transport;

$$\Psi^{(0)}(x, X, T) = e^{i\tilde{\theta}_a} \Psi^{(0)}(X - \xi_a(T), 0). \quad (34)$$

For the discussions below, the Y -integration is done for general N . After a simple Gaussian integral, we get

$$\begin{aligned}\Psi^{(0)}(x, X, T) &= C_b e^{i\theta_b T} \left(\frac{1}{\pi \Delta^2} \right)^{1/4} \sqrt{\frac{Nm}{i \hbar T D}} e^{i\tilde{\theta}_b} \\ &\times \exp \left[-\frac{N^2 m^2 (X - \xi_b(T))^2}{T^2 \hbar^2 2D} + \frac{iNm(X - \xi_b(T))^2}{2\hbar T} \right], \\ D &= \frac{1}{\Delta^2} - \frac{iNm}{\hbar T}.\end{aligned} \quad (35)$$

Taking the absolute square,

$$|\Psi_b^{(0)}(X, T)|^2 = |C_b|^2 \sqrt{\frac{1}{\pi \Delta^2 \rho}} \exp \left[-\frac{(X - \xi_b(T))^2}{\Delta^2 \rho} \right] \quad (36)$$

Here we have written

$$DD^* = \frac{N^2 m^2}{\hbar^2 T^2} \rho, \quad \rho \equiv 1 + \frac{T^2 \hbar^2}{\Delta^4 N^2 m^2}. \quad (37)$$

The peak of $|\Psi^{(0)}(b, X, T)|^2$ traces the classical trajectory $X = \xi_b(T)$ and for large N , $\rho = 1 + O(1/N^2)$, so the effect of the broadening of the width due to the fluctuation of X , is $O(1/N^2)$ so setting $\rho = 1$ is may be allowed. See below for the numerical study.

Now, in order to map microscopically different channels $|a\rangle \neq |b\rangle$ into a macroscopically distinguishable state, Δ has to be sufficiently small compared with the distance of the different peaks; $|\xi_a(T) - \xi_b(T)| \gg \Delta$. This is the requirement for the detector, which is assumed to be the case. To get the numerical value of T required for producing a signal, we estimate

$$|\xi_a(T) - \xi_b(T)| \sim \frac{f_a T^2}{2m} \gg \Delta.$$

Thus we get $T \gg \sqrt{2m\Delta/f_a}$. Let a be the atomic scale length, then $f_a a = f \lambda_a a \sim f O a$ is of the atomic energy

size (ionization energy, for instance) due to the interaction between the object and one particle in the detector. If we set rather arbitrarily $f_a a = 1 \text{ eV} = 1.6 \times 10^{-12} \text{ erg}$, $\Delta = 10^{-3} \text{ cm}$ and take $a = 10^{-7} \text{ cm}$, then for the case of the electron ($m = 9.1 \times 10^{-28} \text{ g}$),

$$T \gg \sqrt{\frac{2m\Delta a}{f_a a}} \sim 10^{-12} \text{ sec.} \quad (38)$$

This is quite a small number, which does not change much for the proton ($m = 1.7 \times 10^{-24} \text{ g}$) and for somewhat larger or smaller Δ .

Next, we try to estimate the magnitude of N , for which the diffusion process during the measurement time T can be neglected. By (36), $\rho - 1 \ll 1$, or equivalently $N \gg \hbar T / (\Delta^2 m)$ has to be satisfied. This follows also from the uncertainty relation. If we adopt $\Delta = (10^{-3}, 10^{-4}) \text{ cm}$, then we get $N \gg (10^6, 10^8) T$ for the electron, $N \gg (10^3, 10^5) T$ for the proton. (T is measured in sec.) Since T given in (38) is quite small, we conclude that the diffusion effect in the detection process is totally negligible. We set $\rho = 1$ hereafter. When we include higher orders of $[H_O]^{\text{nd}}$, non-trivial constraint on N will emerge, see Sec. IV H.

F. The signal function in zeroth order

$J^{(0)}(X, T)$ is given in (27). In the limit \Rightarrow , we get the ideal situation in the measurement. Denoting by \longrightarrow the time evolution after the detector is switched on, we get

$$J^{(0)}(X, 0) = \sum_b |C_b|^2 \delta(X) = \delta(X) \quad (39)$$

$$\longrightarrow J^{(0)}(X, T) = \sum_b |C_b|^2 \delta(X - \xi_b(T)) \quad (40)$$

Note that the signal function becomes the classical density of a point particle moving along $\xi_b(T)$. By ideal, we mean that the above result is in conformity with the usual the quantum mechanical rule; integrating by X in the small region $R_b = (\xi_b(T) - \delta, \xi_b(T) + \delta)$, ($\delta > 0$),

$$\int_{R_b} J^{(0)}(X, t) dX = |C_b|^2. \quad (41)$$

We say that it is the probability for X to stay in R_b , which in turn implies that the probability of the object to be in the state $|b\rangle$ is $|C_b|^2$, since the mapping $|b\rangle \leftrightarrow \xi_b(T)$ is one to one by the stationary phase mechanism.

Written by the wave function symbolically, the ideal measuring process is expressed as

$$\begin{aligned} \Psi(x, X, 0) &= \sum_a C_a \langle x|a\rangle \sqrt{\delta(X)} \longrightarrow \\ \Psi(x, X, T) &= \sum_a C_a \langle x|a\rangle e^{i(\theta_a T + \bar{\theta}_a)} \sqrt{\delta(X - \xi_a(T))}. \end{aligned}$$

Thus the object stays in the same state. (Square-root of the delta-function is ill-defined so we need some regularization.)

IV. HIGHER ORDERS

A. $\Psi^{(1)}(b, X, T)$

As is given in (22), $\langle X|U_{N,b}(T-s)U_{N,a}(s)|Y\rangle$ has to be evaluated. This is the evolution kernel defined by the Hamiltonian $H(t) = Nm\dot{X}^2/2 - Nf(t)X$, where

$$f(t) = \begin{cases} f_a; & \text{for } 0 < t < s, \\ f_b; & \text{for } s < t < T. \end{cases} \quad (42)$$

Now we apply the formula for this process [20]

$$\langle Y|U_{N,b}(T-s)U_{N,a}(s)|X\rangle = \sqrt{\frac{Nm}{2\pi i \hbar T}} \exp\left(\frac{i}{\hbar} N S_{ba}\right),$$

$$\begin{aligned} S_{ba} &= \frac{m(Y-X)^2}{2T} \\ &+ \frac{Y}{T} \int_0^T dt f(t)t + \frac{X}{T} \int_0^T dt f(t)(T-t) \\ &- \frac{1}{Tm} \int_0^T dt \int_0^T dt' f(t)f(t')(T-t)t'. \end{aligned}$$

Using (42), we get after a straightforward calculation,

$$S_{ba} = \frac{m}{2T} \left(Y - X - \xi_{ba}(T, s) \right)^2 \quad (43)$$

$$+ X Q_{ba}(T, s) + P_{ba}(T, s),$$

$$Q_{ba}(T, s) = f_a s + f_b (T - s), \quad (44)$$

$$\begin{aligned} P_{ba}(T, s) &= -\frac{f_a^2}{6m} s^2 (3T - 2s) \\ &- \frac{f_b^2}{6m} (T - s)^3 - \frac{f_a f_b}{2m} s (T - s)^2, \end{aligned} \quad (45)$$

$$\xi_{ba}(T, s) = \frac{f_a}{2m} (2Ts - s^2) + \frac{f_b}{2m} ((T - s)^2). \quad (46)$$

Here, $\xi_{ba}(T, s)$ is the classical change of X during T under the force $f(t)$. Note that $\xi_{ba}(T, 0) = \xi_b(T)$ and $\xi_{ba}(T, T) = \xi_a(T)$, in conformity with the fact that s is the time H_O acted, making the transition from $|a\rangle$ to $|b\rangle$. One can confirm again that $P_{ba}(T, s) + X Q_{ba}(T, s)$ coincides with the classical action integral along $\xi_{ba}(T, s)$. Using (12), (24) and applying the operation $\int dY \sum_a C_a \langle Y|\Phi\rangle$, we finally obtain

$$\Psi^{(1)}(b, X, T) \quad (47)$$

$$\begin{aligned} &= \left(\frac{1}{\pi \Delta^2} \right)^{1/4} \sqrt{\frac{Nm}{2\pi i \hbar T}} \sqrt{\frac{2\pi}{D}} \sum_a (-i/\hbar) (H_O)_{ba} C_a \\ &\times \int_0^T ds \times \exp i\{\theta_a s + \theta_b (T - s)\} \\ &\times \exp \left[-\frac{N^2 m^2 R_{ba}(X, T, s)^2}{2\hbar^2 T^2 D} + \frac{N m i R_{ba}(X, T, s)^2}{2\hbar T} \right. \\ &\quad \left. + \frac{i}{\hbar} N \{X Q_{ba}(T, s) + P_{ba}(T, s)\} \right], \end{aligned} \quad (48)$$

$$R_{ba}(X, T, s) = X - \xi_{ba}(T, s), \quad (49)$$

Note that $R_{ba}(X, T, 0) = X - \xi_b(T)$.

B. Signal function $J^{(1)}(X, T)$

By (28), (35) and (48), one obtains

$$J^{(1)}(X, T) = \left(\frac{1}{\pi \Delta^2 \rho} \right)^{1/2} (-i/\hbar) \sum_{ab} C_b^*(H_O)_{ba} C_a \\ \times \int_0^T ds \times \exp i\{(\theta_a - \theta_b)s\} \exp \Phi_{ba} + \text{c.c.}$$

The explicit expression of Φ_{ba} is shown in (B4), (B5) of Appendix B; The result is

$$\Phi_{ba} = -\frac{1}{\Delta^2} \left(X - \frac{\xi_{ba}(T, s) + \xi_{ba}(T, 0)}{2} \right)^2 \\ - \frac{1}{4\Delta^2} (\xi_{ba}(T, s) - \xi_{ba}(T, 0))^2 + \frac{iN}{\hbar} \omega_{ba}(X, T, s). \quad (50)$$

Here $\omega_{ba}(X, T, s)$ is given by

$$\omega_{ba}(X, T, s) = X(f_a - f_b)s - \frac{f_a^2}{6m} s^2 (3T - 2s) \\ - \frac{f_b^2}{6m} \{(T - s)^3 - T^3\} - \frac{f_a f_b}{2m} s(T - s)^2.$$

When $N \rightarrow \infty$, the integration by s is dominated by the stationary point, satisfying

$$0 = \frac{d\omega_{ba}(X, T, s)}{ds} = (f_a - f_b) \left(X - \frac{f_b T^2}{2m} \right. \\ \left. - \frac{(f_a - f_b)Ts}{m} + \frac{(2f_a - f_b)s^2}{2m} \right). \quad (51)$$

Taking in advance the limit $\Delta \rightarrow 0$ into consideration,

$$X - \frac{\xi_{ba}(T, s) + \xi_{ba}(T, 0)}{2} = 0, \quad (52)$$

$$\xi_{ba}(T, s) - \xi_{ba}(T, 0) = (f_a - f_b)(2T - s)s = 0 \quad (53)$$

have to be satisfied also. The only solution of (53) in the range $0 \leq s \leq T$ is $s = 0$, which also satisfies (51). Then (52) becomes $X - f_b T^2/2m = 0$. Thus, as a function of X , $\Psi^{(0)*}\Psi^{(1)}$ has a peak at $X = \xi_b(T)$. These facts are expected; suppose the peak of $\Psi^{(0)}(X, T)$ is at $\xi_b(T)$. In order to get non-zero $J^{(1)}(X, T)$, the peak of $\Psi^{(1)}(X, T)$ should also be at $\xi_b(T)$. This can be realized if and only if the transition caused by $[H_O]^{\text{nd}}$ from the state $|a\rangle$ to $|b\rangle$ occurs at $s = 0$. Then the X -integration in $\Psi^{(0)*}\Psi^{(1)}$ is dominated near $X = \xi_b(T)$. To perform the calculation, consider the region near $s = 0$, $X = \xi_b(T)$;

$$0 = \frac{d\omega_{ba}(X, T, s)}{ds} \\ \sim (f_a - f_b) \left(X - \frac{f_b T^2}{2m} - \frac{f_a - f_b}{m} Ts \right) + O(s^2). \quad (54)$$

Thus, near $s = 0$, the stationary trajectory $s = s(X)$ and the second derivative (which is X -independent) becomes

$$s = s(X) = \frac{m(X - \xi_b(T))}{(f_a - f_b)T} + O((X - \xi_b(T))^2), \quad (55)$$

$$\frac{\partial^2 \omega_{ba}(X, T, s)}{\partial s^2} = -\frac{(f_a - f_b)^2 T}{m}. \quad (56)$$

Note that the second derivative is a constant. Keeping X fixed, s -integration is first performed by expanding $\omega_{ba}(X, T, s)$ around $s = s(X)$

$$\omega_{ba}(X, T, s) = \omega_{ba}(X, T, s(X)) \\ - (T/2m)(f_a - f_b)^2 (s - s(X))^2 + \dots$$

In this way, we get

$$\int_0^T ds \exp(iN\omega_{ba}(X, T, s)/\hbar) = \exp(iN\omega_{ba}(X, T, s(X))/\hbar) \\ \times \sqrt{\frac{-2i\pi\hbar m}{N(f_a - f_b)^2 T}} \left(1 + O(1/\sqrt{N}) \right). \quad (57)$$

The above expression is a function of X . Now we consider its asymptotic functional form when $N \rightarrow \infty$. The stationary point is given by

$$0 = \frac{d\omega_{ba}(X, T, s(X))}{dX} = (f_a - f_b)s(X).$$

Therefore $X = \xi_b(T)$ and

$$\omega_{ba}(X = \xi_b(T), T, s(\xi_b(T))) = \omega_{ba}(\xi_b(T), T, 0) = 0, \\ \frac{d^2 \omega_{ba}(T, s(X))}{dX^2} = (f_a - f_b) \frac{ds(X)}{dX} = \frac{m}{T}.$$

Thus for large N , one can write

$$\exp(iN\omega_{ba}(X, T, s(X))/\hbar) \\ = \exp\left(\frac{iN}{\hbar}\right) \left(\frac{m}{2T} (X - \xi_b(T))^2 + O((X - \xi_b(T))^3) \right), \\ \Rightarrow \frac{1}{2} \sqrt{\frac{2i\pi\hbar T}{Nm}} \left(\delta(X - \xi_b(T)) + O(1/\sqrt{N}) \right). \quad (58)$$

The factor $1/2$ in front appears by the following reason. By (55) and by $s > 0$, it follows that $X > \xi_b(T)$ ($X < \xi_b(T)$) if $f_a > f_b$ ($f_a < f_b$) along the stationary trajectory. Therefore, in either case, $X = \xi_b(T)$ is the end point of the X -integration and using the formula $\int_0^\infty dx \delta(x) = 1/2$, eq. (58) follows.

Other factors in (50) not multiplied by N are unity for large N , when the stationary value is inserted. In fact, consider

$$\exp\left\{ -\frac{1}{\Delta^2} \left(X - \frac{\xi_{ba}(T, s(X)) + \xi_{ba}(T, 0)}{2} \right)^2 \right\} \quad (59)$$

$$\times \exp\left\{ -\frac{1}{4\Delta^2} (\xi_{ba}(T, s(X)) - \xi_{ba}(T, 0))^2 \right\}. \quad (60)$$

Since (58) says that $X - \xi_b(T) = O(1/\sqrt{N})$ and $s(X) \sim X - \xi_b(T)$, one can estimate for large N as

$$\xi_{ba}(T, s(X)) - \xi_{ba}(T, 0) = O(s(X)) \\ = O(X - \xi_b(T)) = O(1/\sqrt{N}), \\ X - \frac{\xi_{ba}(T, s(X)) + \xi_{ba}(T, 0)}{2} = X - \xi_b(T) \\ - \xi'_{ba}(T, 0)s(X)/2 - \xi''_{ba}(T, 0)s(X)^2/4 + \dots \\ = -\xi''_{ba}(T, 0)s(X)^2/4 + \dots$$

This is $O(X - \xi_b(T))^2 = O(1/N)$. Therefore, both factors of (59) and (60) becomes unity as $N \rightarrow \infty$.

Collecting (57), (58), and adding the term with complex conjugate, we arrive at

$$\begin{aligned} J^{(1)}(X, T) &= \sum_b \Psi_b^{(0)*}(X, T) \Psi_b^{(1)}(X, T) + \text{c.c.} \\ &= \frac{1}{N} \sum_b K_b^{(1)} \delta\left(X - \frac{f_b^2 T}{2m}\right), \end{aligned} \quad (61)$$

$$K_b^{(1)} = \frac{\sqrt{\pi}}{\Delta} \sum_a \frac{2\text{Im} C_b^*(H_O)_{ba} C_a}{|f_a - f_b|} + O(1/N). \quad (62)$$

This is the first order correction in $[H_O]^{\text{nd}}$ to the ideal case (40). In Appendix D, the result (62) is checked by integrating over X first and then by s . Note that \sqrt{T} in (57) and (58) are cancelled, so K_b is independent of T for each b . In this connection, see Sec.IV G.

C. Normalization

The normalization $\int dX J(X, T) = 1$ leads to

$$\begin{aligned} \int dX J^{(0)}(X, T) &= 1, \\ \int dX J^{(k)}(X, T) &= 0 \quad (k = 1, 2, \dots). \end{aligned} \quad (63)$$

We can check $\int dX J^{(1)}(X, T) = 0$. Indedd, note that

$$\int dX J^{(1)}(X, T) = \frac{\sqrt{\pi}}{N\Delta} \sum_{a,b} \frac{2\text{Im} C_b^*(H_O)_{ba} C_a}{|f_a - f_b|}.$$

Here, $1/|f_a - f_b|$ is real and symmetric under $a \leftrightarrow b$. Then, we see that $\sum_{b \neq a} C_b^*(H_O)_{ba} C_a / |f_a - f_b|$ is a real number, so the imaginary part vanishes.

D. Calculation of $J_1^{(2)}(X, T)$

Consider $J_1^{(2)}$ of (29). It is expressed by

$$\begin{aligned} J_1^{(2)}(X, T) &= \sum_b \Psi_b^{(1)*}(X, T) \Psi_b^{(1)}(X, T) \\ &= \left(\frac{1}{\pi\Delta^2\rho}\right)^{1/2} \sum_{aa'b} C_{a'}^*(H_O)_{a'b} (H_O)_{ba} C_a \int_0^T ds' \int_0^T ds \\ &\quad \times \exp i\{-(\theta_{a'} s' + \theta_b(s - s') - \theta_a s)\} \exp \Phi_{a'a;b}. \end{aligned} \quad (64)$$

$\Phi_{a'a;b}$ is given in (C2) and (C3) of Appendix C;

$$\begin{aligned} \Phi_{a'a;b} &= -\frac{1}{\Delta^2} \left(X - \frac{\xi_{ab}(T, s) + \xi_{a'b}(T, s')}{2}\right)^2 \\ &\quad - \frac{1}{4\Delta^2} (\xi_{ba}(T, s) - \xi_{ba'}(T, s'))^2 + \frac{iN}{\hbar} \omega_{aa';b}(T, s), \end{aligned} \quad (65)$$

$$\begin{aligned} \omega_{a'a;b}(T, s) &= \omega_{ba}(T, s) - \omega_{ba'}(T, s') \\ &= X(Q_{ba}(T, s) - Q_{ba'}(T, s')) + P_{ba}(T, s) - P_{ba'}(T, s'). \end{aligned} \quad (66)$$

The stationary equation in s is identical to (51);

$$0 = \frac{d\omega_{aa';b}(T, s, s')}{ds} = \frac{d\omega_{ba}(T, s)}{ds} \quad (67)$$

The solution is written as $s = s_{ba}(X)$. Similarly, we have

$$0 = \frac{d\omega_{aa';b}(T, s, s')}{ds'} = -\frac{d\omega_{ba'}(T, s')}{ds'} \quad (68)$$

$$= -(f_{a'} - f_b) \left(X - \frac{f_b T^2}{2m} - \frac{(f_{a'} - f_b) T s'}{m} - \frac{(2f_{a'} - f_b) s'^2}{2m} \right), \quad (69)$$

with the solution $s' = s'_{ba'}(X)$. Now $\Phi_{aa';b}(X, s, s')$ is expanded around $s = s_{ba}(X)$ and $s' = s'_{ba'}(X)$, and we calculate the second derivative at these points,

$$\begin{aligned} M_{ba} &\equiv \partial^2 \omega_{aa';b} / \partial s^2 = -(f_a - f_b)^2 T / m \\ &\quad - (f_a - f_b)(2f_a - f_b) s_{ba}(X) / m, \end{aligned} \quad (70)$$

$$\begin{aligned} M_{ba'} &\equiv \partial^2 \omega_{aa';b} / \partial s'^2 = (f_{a'} - f_b)^2 T / m \\ &\quad + (f_{a'} - f_b)(2f_{a'} - f_b) s'_{ba'}(X) / m. \end{aligned} \quad (71)$$

The reult of s, s' integration is

$$\begin{aligned} &\int_0^T ds \int_0^T ds' \exp i\{-(\theta_{a'} s' + \theta_b(s - s') - \theta_a s)\} \\ &\quad \times \exp \Phi_{aa';b}(X, s, s') \\ &= \sqrt{\frac{(2\pi)^2 \hbar^2}{N^2 M_{ba} M_{ba'}}} \cdot \exp \Phi_{aa';b}(X, s_{ba}(X), s'_{ba'}(X)). \end{aligned} \quad (72)$$

Next task is to study the X -integration. For that purpose, it is convenient to use the following form for the factor appearing in (65);

$$\begin{aligned} P &\equiv \exp \left[-\frac{1}{\Delta^2} \left(X - \frac{\xi_{ba}(T, s) + \xi_{ba'}(T, s')}{2} \right)^2 \right] \\ &\quad \times \exp \left[-\frac{1}{4\Delta^2} (\xi_{ba}(T, s) - \xi_{ba'}(T, s'))^2 \right] \\ &= \exp \left[-\frac{1}{2\Delta^2} \left(X - \xi_{ba}(T, s) \right)^2 \right] \\ &\quad \times \exp \left[-\frac{1}{2\Delta^2} \left(X - \xi_{ba'}(T, s') \right)^2 \right]. \end{aligned} \quad (73)$$

Due to the structure of $\Phi_{aa';b}(X, s_{ba}(X), s'_{ba'}(X))$, the resulting dependence on X differs for $a' = a$ and $a' \neq a$.

The case $a = a'$

$\omega_{aa;b}(X, s_{ba}(X), s'_{ba'}(X)) = 0$ holds since for $a = a'$ $s_{ba}(X) = s'_{ba'}(X) \equiv s(X)$. Thus the result of the X -integration is a constant independent of N . Consider the factor (73) contained in $\Phi_{aa';b}$ of (65). Inserting the stationary value $s(X) = s_{ba}(X)$ in the first factor of the

right-hand side of (73), we concentrate on $X - \xi_{ba}(s(X))$. The factor in the second parenthesis in (51) is rearranged as

$$\begin{aligned} X - \frac{1}{2m} \left(f_b T^2 + 2(f_a - f_b)Ts(X) - (2f_a - f_b)s(X)^2 \right) \\ = X - \xi_{ba}(T, s(X)) + \frac{1}{2m} f_a s(X)^2. \end{aligned} \quad (74)$$

In this way, we get

$$X - \xi_{ba}(T, s(X)) = -(f_a/2m)s(X)^2. \quad (75)$$

Since we are considering $\Delta \rightarrow 0$ (after $N \rightarrow \infty$), $X - \xi_{ba}(T, s(X)) \rightarrow 0$, implying $s(X) \rightarrow 0$. On the other hand, by (46), one can approximate

$$\begin{aligned} X - \xi_{ba}(T, s(X)) \\ = X - \frac{f_b T^2}{2m} - \frac{(f_a - f_b)T}{m} s(X) = 0. \end{aligned}$$

Solving this relation for $s(X)$ and inserting it back into (75), we conclude

$$X - \xi_{ba}(T, s(X)) = -\frac{f_a}{2m} \frac{m^2}{(f_a - f_b)^2 T^2} \left(X - \frac{f_b T^2}{2m} \right)^2.$$

Thus we obtain

$$\begin{aligned} P &= \exp \left[-\frac{C_{ba}}{\Delta^2 T^4} \left(X - \frac{f_b T^2}{2m} \right)^4 \right], \\ C_{ba} &= \frac{(f_a^2}{4m^2} \frac{m^4}{(f_a - f_b)^4}. \end{aligned}$$

Here the following formula is adopted. With $C > 0$,

$$\lim_{\Delta \rightarrow 0} \exp \left[-C \frac{(X - a)^4}{\Delta^2} \right] = \frac{\gamma \sqrt{\Delta}}{C^{1/4}} \delta(X - a),$$

where $\gamma \equiv \int_{-\infty}^{\infty} dz \exp(-z^4)$. In this way, we arrive at

$$P = \frac{\gamma \sqrt{\Delta}}{C^{1/4}} \frac{1}{2} \delta \left(X - \frac{f_b T^2}{2m} \right).$$

The factor $1/2$ is present for the same reason as given concerning (58). Collecting all factors, the result for $J_{1;a=a'}^{(2)}$ is obtained as follows. In doing so, $s_{ba}(X)$ and $s_{ba'}(X)$ appearing in M_{ba} , $M_{ba'}$ of (70), (71) can be set to zero, which is inserted into (72). We use (37) and set $\rho = 1$.

$$\begin{aligned} J_{1;a=a'}^{(2)} &= \sum_{a=a'b} \delta_{aa'} (-i/\hbar) (i/\hbar) C_{a'}^* (HO)_{a'b} (HO)_{ba} C_a \\ &\times \sqrt{\frac{1}{\pi \Delta^2}} \frac{Nm}{2\pi \hbar T} \frac{2\pi}{\sqrt{DD^*}} \\ &\times \frac{1}{N} \frac{\pi \hbar 2m}{(f_a - f_b)^2 T} \frac{\gamma \sqrt{\Delta} T}{C_{ab}^{1/4}} \frac{1}{2} \delta \left(X - \frac{f_b T^2}{2m} \right) \\ &= \frac{1}{N} \sum_b K_b^{(2)} \delta \left(X - \frac{f_b T^2}{2m} \right), \\ K_b^{(2)} &= \frac{\sqrt{2} \gamma \sqrt{m \pi}}{\sqrt{\Delta} \hbar} \sum_a \frac{C_a^* (HO)_{ab} (HO)_{ba} C_a}{|f_a - f_b| \sqrt{|f_a|}}. \end{aligned} \quad (76)$$

The case $a \neq a'$

In this case, $s'(X) \neq s(X)$ so $\Phi_{a'a;b}(X, s(X), s'(X))$ does not vanish and is a function of X of $O(N)$, producing a stationary phase. We need to pin down the position. (We write $s_{ba}(X) = s(X)$ and $s'_{bab}(X) = s'(X)$.) By the stationarity in s and s' , eq.(75) and a similar equation for s' with a replaced by a' hold;

$$X - \xi_{ba'}(T, s'(X)) = -\frac{f_{a'} T^2}{2m} s'(X)^2. \quad (77)$$

Eqs.(75) and (77) assure that we can limit our discussions near $s(X) = 0$ and $s'(X) = 0$, as stated just below (75).

Now, using the stationarity in s and s' , the stationary condition of $\Phi_{a'a;b}(X, s(X), s'(X))$ with respect to X can be written as,

$$\begin{aligned} 0 &= \frac{d\Phi_{a'a;b}(X, s(X), s'(X))}{dX} = \frac{\partial \Phi_{a'a;b}(X, s(X), s'(X))}{\partial X} \\ &= (f_a s(X) + f_b(T - s(X)) - ((f_{a'} s'(X) + f_b(T - s'(X)))) \\ &= (f_a - f_b)s(X) - (f_{a'} - f_b)s'(X). \end{aligned}$$

Higher derivatives are obtained by differentiating the stationary equation of s or s' . Differentiate (51) by X ;

$$\begin{aligned} 1 - 2\frac{1}{2m}(f_a - f_b)\frac{ds}{dX}T + \frac{1}{2m}(2f_a - f_b)2s\frac{ds}{dX} &= 0, \\ -2\frac{1}{2m}(f_a - f_b)\frac{d^2s}{dX^2}T & \\ + \frac{1}{2m}(2f_a - f_b)2 \left\{ s\frac{d^2s}{dX^2} + \left(\frac{ds}{dX} \right)^2 \right\} &= 0. \end{aligned}$$

Setting $s = 0$ in the first equation, $(f_a - f_b)(ds/dX) = m/T$, which is inserted into the second. Thus one gets

$$(f_a - f_b)\frac{d^2s}{dX^2} = \frac{m^2}{T^3} \frac{(2f_a - f_b)}{(f_a - f_b)^2}.$$

Using the similar equations for s' , with the replacement $a \rightarrow a'$, we get at $s(X) = 0$, $s'(X) = 0$,

$$\begin{aligned} \frac{d\Phi_{a'a;b}}{dX} &= 0, \quad \frac{d^2\Phi_{a'a;b}}{dX^2} = 0, \\ \frac{1}{3!} \frac{d^3\Phi_{a'a;b}}{dX^3} &= \frac{1}{3!} (f_a - f_b) \frac{d^2s}{dX^2} - (f_{a'} - f_b) \frac{d^2s'}{dX^2} \\ &= \frac{m^2}{6T^3} \frac{(f_a - f_{a'})(f_a f_b + f_{a'} f_b - 2f_a f_{a'})}{(f_a - f_b)^2 (f_{a'} - f_b)^2} \equiv \frac{C_{a'ab}}{T^3}. \end{aligned} \quad (78)$$

In this way, $\Phi_{a'a;b}$ becomes

$$\Phi_{a'a;b} \sim \frac{iN}{\hbar} \frac{C_{a'ab}}{T^3} \left(X - \frac{f_b T^2}{2m} \right)^3. \quad (79)$$

Here we use the formula of symmetric integration

$$\begin{aligned} \lim_{N \rightarrow \infty} \exp i \alpha N (X - a)^3 &= \lim_{N \rightarrow \infty} \cos \alpha N (X - a)^3 \\ &= (|\alpha|N)^{-1/3} \gamma' \delta(X - a), \quad \gamma' = \int_{-\infty}^{\infty} dz \cos^3 z. \end{aligned}$$

Just as in the discussions for $J^{(1)}$ given in (59) and (60), the factors which are independent of N become unity, since $N \rightarrow \infty$ is taken before $\Delta \rightarrow 0$. The result for $a \neq a'$ is thus obtained;

$$\begin{aligned}
J_{1;a \neq a'}^{(2)} &= \sum_{a \neq a'; b'} (-i/\hbar)(i/\hbar) C_{a'}^*(H_O)_{a'b}(H_O)_{ba} C_a \\
&\times \sqrt{\frac{1}{\pi \Delta^2}} \frac{Nm}{2\pi \hbar T} \frac{2\pi}{\sqrt{DD^*}} \\
&\times \frac{1}{N} \frac{\pi \hbar 2m}{|f_a - f_b| |f_{a'} - f_b| T} \frac{\gamma' \hbar^{1/3} T}{C_{aa'b}^{1/3} N^{1/3}} \delta\left(X - \frac{f_b T^2}{2m}\right) \\
&= \frac{1}{N N^{1/3}} \sum_b K_b'^{(2)} \delta\left(X - \frac{f_b T^2}{2m}\right), \\
K_b'^{(2)} &= \frac{2\gamma' m \sqrt{\pi}}{\Delta \hbar} \sum_{a \neq a'} \text{Re} \frac{C_{a'}^*(H_O)_{a'b}(H_O)_{ba} C_a \hbar^{1/3}}{|f_a - f_b| |f_{a'} - f_b| C_{a'ab}^{1/3}}. \quad (80)
\end{aligned}$$

Here Re signifies the real part. Under the exchange $a \leftrightarrow a'$, both $C_{a'}^*(H_O)_{a'b}(H_O)_{ba} C_a$ and $C_{a'ab}^{1/3}$ become complex conjugate. So the summation over all $a \neq a'$ is equivalent to apply Re. The results (76) and (80) have been checked by interating by X first and then by s, s' .

E. Calculation of $J_2^{(2)}$

The remaining term of $O([H_O]^{\text{nd}})^2$ appearing in $J^{(2)}$ is $\Psi^{(0)*}(b, X, T) \Psi^{(2)}(b, X, T) + \text{c.c.}$, see (30). Actually, it is not necessary to calculate this term from the start. One can invoke to the normalization condition (63) with $k = 2$. Consider the result of $J_1^{(2)}$. If we write $J_1^{(2)} = \sum_b J_1^{(2)b}$, then

$$\begin{aligned}
J_1^{(2)b}(X, T) &= \Psi^{(1)*}(b, X, T) \Psi^{(1)}(b, X, T) \\
&= \sum_{a, a'} V_{a'a;b} C_{a'}^*(H_O)_{a'b}(H_O)_{ba} C_a \delta(X - \xi_b(T)). \quad (81)
\end{aligned}$$

$V_{a'a;b}$ is given by the sum of (76) and (80) and has different forms for $a = a'$ and $a \neq a'$. As dictated from the definition of $U^{(2)}$ of (23), we can write $J_2^{(2)} = \sum_b J_2^{(2)b}$, where $J_2^{(2)b}$ has the form

$$\begin{aligned}
J_2^{(2)b}(X, T) &= \Psi^{(0)*}(b, X, T) \Psi^{(2)}(b, X, T) + \text{c.c.} \\
&= \sum_{c, a} W_{ba;c} C_b^*(H_O)_{bc}(H_O)_{ca} C_a \delta(X - \xi_b(T)). \quad (82)
\end{aligned}$$

Note the difference of the index structure of b between (81) and (82). Inserting these two into (63) ($k = 2$),

$$\begin{aligned}
&\sum_{c, a, b} W_{ba;c} C_b^*(H_O)_{bc}(H_O)_{ca} C_a \\
&= - \sum_{a, a', b} V_{a'a;b} C_{a'}^*(H_O)_{a'b}(H_O)_{ba} C_a
\end{aligned}$$

is obtained. After renaming the index of W , and recalling that the above equation holds for any C_a and any

operator H_O , $W_{a'a;b} = -V_{a'a;b}$ follows. Thus we arrive at for $a = a'$ and $a \neq a'$ separately;

$$\begin{aligned}
J_{2;a=a'}^{(2)}(X, T) &= -\frac{1}{N} \frac{\sqrt{2}\gamma\sqrt{m\pi}}{\sqrt{\Delta}\hbar} \\
&\times \sum_{a, b} \frac{C_a^*(H_O)_{ab}(H_O)_{ba} C_a}{|f_a - f_b| \sqrt{f_a}} \delta(X - \xi_a(T)), \quad (83)
\end{aligned}$$

$$\begin{aligned}
J_{2;a \neq a'}^{(2)}(X, T) &= -\frac{1}{N N^{1/3}} \frac{2\gamma' m \sqrt{\pi}}{\Delta \hbar} \\
&\times \sum_{a \neq a', b} 2\text{Re} \frac{C_{a'}^*(H_O)_{a'b}(H_O)_{ba} C_a \hbar^{1/3}}{|f_{a'} - f_b| |f_a - f_b| C_{a'ab}^{1/3}} \delta(X - \xi_{a'}(T)). \quad (84)
\end{aligned}$$

F. The result for $J(X, T)$ up to $([H_O]^{\text{nd}})^2$

Collecting all the results of (40), (62), (76), (80) (83) and (84), the signal function up to $O([H_O]^{\text{nd}})^2$ is

$$\begin{aligned}
J(X, T) &= J^{(0)}(X, T) + J^{(1)}(X, T) + J_{1;a=a'}^{(2)}(X, T) \\
&\quad + J_{1;a \neq a'}^{(2)}(X, T) + J_{2;a=a'}^{(2)}(X, T) + J_{2;a \neq a'}^{(2)}(X, T) \\
&= \sum_b \left[|C_b|^2 + \frac{A_b}{N} + \frac{B_b}{N N^{1/3}} \right] \delta(X - f_b T^2 / 2m). \quad (85)
\end{aligned}$$

Here each coefficient is given by

$$\begin{aligned}
A_b &= A_{b1} + A_{b2}, \\
A_{b1} &= \frac{\sqrt{\pi}}{\Delta} \sum_a \frac{2\text{Im} C_b^*(H_O)_{ba} C_a}{|f_a - f_b|}, \quad (86)
\end{aligned}$$

$$\begin{aligned}
A_{b2} &= \frac{\sqrt{2}\gamma\sqrt{m\pi}}{\sqrt{\Delta}\hbar} \left(\sum_a \frac{C_a^*(H_O)_{ab}(H_O)_{ba} C_a}{|f_a - f_b| \sqrt{|f_a|}} \right. \\
&\quad \left. - \sum_c \frac{C_b^*(H_O)_{cb}(H_O)_{ca} C_b}{|f_b - f_c| \sqrt{|f_a|}} \right), \quad (87)
\end{aligned}$$

$$\begin{aligned}
B_b &= \frac{2\gamma' m \sqrt{\pi} \hbar^{1/3}}{\Delta \hbar} \\
&\times \text{Re} \left(\sum_{a \neq a'} \frac{C_{a'}^*(H_O)_{a'b}(H_O)_{ba} C_a}{|f_a - f_b| |f_{a'} - f_b| C_{a'ab}^{1/3}} \right. \\
&\quad \left. - \sum_{b \neq a, c} \frac{C_b^*(H_O)_{bc}(H_O)_{ca} C_a}{|f_b - f_c| |f_a - f_c| C_{bac}^{1/3}} \right). \quad (88)
\end{aligned}$$

The origin of the power of $1/N$ is the stationary phase integrations by s, s' and X . Each s -integration brings us $1/\sqrt{N}$, but the results of the X -integration (or equivalently of the formula (3)) depend on the situation; $1/\sqrt{N}$ for A_{b1} , $(1/\sqrt{N})^0$ for A_{b2} (no stationary phase) and for B_b , $1/N^{1/3}$. In Appendix E, the order of neglected terms are estimated. They are shown to be down by at least $1/\sqrt{N}$ compared with (85).

As $T \rightarrow 0$, above $J(X, T)$ goes over to $J(X, 0) = \delta(X)$ by the same relation that follows from the normalization condition. Note further that when $[H_O, O] = 0$, all the correction terms are absent and the ideal result becomes exact up to the order considered here.

G. T-independence of $A_{b1,b2}$ and B_b

Above correction terms A_{b1} , A_{b2} and B_b are independent of T , besides $|C_b|^2$. The reason is simple; as we have seen above, the fluctuation into any channel b occurs at $t = 0$ for $N \rightarrow \infty$, and after that the state develops by $U_{N,b}$ of (17) and (18), so the classical time evolution is realized. Consider the signal function $J(b, X, T) = |\Psi(b, X, T)|^2$ defined for each channel, i.e. one term in the sum (25) of $J(X, T) = \sum_b J(b, X, T)$. Then once the above fluctuation at $t = 0$ is taken into account, $J(b, X, T)$ evolves as $\delta(X - \xi_b(T))$ without any further T dependence. Therefore, when we integrate $J(b, X, T)$ in R_b of (41) and define

$$J(b, T) \equiv \int_{R_b} J(b, X, T) = \int_{R_b} J(X, T) dX,$$

then $J(b, T)$ is independent of T . Since this holds for any H_O or for any parameters in the theory, we conclude that A_{b1} , A_{b2} or B_b does not depend on T . This statement applies even if one includes higher order of H_O . However, when the diffusion process is taken into account, the above assertion does not hold.

H. Numerical estimates

Here we try to estimate the order of numerical values of the correction terms. Consider first A_{b1} given in (86); The numerator is the order of energy of the object. As discussed in deriving (38), $f_a a$ order of the energy a particle in the detector receives from the object. Thus we regard $C_b^*(H_O)_{ba} C_a$ is of the same order as $|f_a - f_b|a \sim f_a a$. In this way, adopting $\Delta = 10^{-3}\text{cm}$, $a = 10^{-7}\text{cm}$,

$$\frac{A_{b1}}{N} \sim \frac{1}{N} \times \frac{a}{\Delta} \sim \frac{1}{N} \times 10^{-4}. \quad (89)$$

The condition $a/N\Delta \ll 1$ is our requirement for the measuring process to be sensible. Thus one has to require $N \gg 10^4$. We have stressed throughout the paper that the limit $N \rightarrow \infty$ is taken before $\Delta \rightarrow 0$, which is denoted by \Rightarrow . The requirement $a/N\Delta \ll 1$ agrees with this limit and at the same time gives the precise condition of this limit. In our case, the ‘‘macroscopic’’ number of particles in the detector required to produce experimental signal is expressed as $N \gg 10^4$, which can be much smaller than, for example, the Avogadro number. The last conclusion is stated frequently for realistic devices.

Next, we estimate A_{b2} of (87). Similarly as above, the factor $C_b^*(H_O)_{cb}(H_O)_{ca}C_b$ is regarded as the same order with $|f_a - f_b|f_a a^2 \sim f_a^2 a^2$. Then apart from the numerical constant, $A_{b2} \sim (a^2/\hbar)\sqrt{|f_a|m/\Delta}$. In this way,

$$\frac{A_{b2}}{N} \sim \frac{1}{N} \sqrt{\frac{|f_a|a}{\hbar^2/(ma^2)}} \left(\frac{a}{\Delta}\right)^{1/2} \sim \frac{1}{N} \times 4.0 \times 10^{-2}. \quad (90)$$

Here we have used the same values as in (38). For the electron, $\hbar^2/(ma^2) \sim 10^{-13}\text{erg}$. The above (90) is much

larger than (89) and may be the main correction to the ideal zeroth order result of (85). The reason why this term is large is, as stated before, understood if we look at (65). When $a = a'$, $\omega_{aa;b}(X, s_{ab}(X), s'_{ab}(X)) = 0$; after inserting the stationary condition of s and s' , the exponent of Φ becomes $O(1)$, not $O(N)$, so the cancellation of the phase as a function of X does not occur and the resulting value becomes large. When, for example, $N = 10^5$ particles are participating in giving the signal, the correction of the order 10^{-7} is expected. This can be within the experimental confirmation.

Finally as for B_b , we note $C_{bac}/\hbar \sim m^2/f_a$ by (78). In this way, we get

$$\begin{aligned} \frac{B_b}{NN^{1/3}} &= \frac{1}{NN^{1/3}} \frac{ma^2}{\hbar} \times \frac{1}{(C_{abc}/\hbar)^{1/3}} \times \frac{a}{\Delta} \\ &= \frac{1}{NN^{1/3}} \left(\frac{f_a a}{\hbar^2/(ma^2)}\right)^{1/3} \frac{a}{\Delta} \sim \frac{1}{NN^{1/3}} \times 2.5 \times 10^{-4}. \end{aligned}$$

This is a small number.

V. DISCUSSIONS

In order to see whether or not our result (85) is specific to our model, let us consider the general Hamiltonian (5), combined with (6) and apply the arguments of Appendix A. When $N \rightarrow \infty$, the leading term agrees with the ideal result of (85). This is because H_O is negligible and the object stays in the prescribed eigen-state $|b\rangle$ of O . For fixed b , the detector evolves by $H_D + H_I(\lambda_b, [x_i])$, so the stationary path $X_b^{\text{st}}(t)$ depends on b . By normalization, the signal function (A2) applied at t takes the form

$$J(X, t) = \sum_b |C_b|^2 \delta(X - X_b^{\text{st}}(t))$$

for general case also. Expanding in $[H_O]^{\text{nd}}$, we use (14), with one s integration accompanying each H_O . Since each s integral is dominated by the stationary phase, one s brings us $1/\sqrt{N}$, irrespective of the form of h and g . We conclude that k -th order correction terms contains at least the factor $(1/\sqrt{N})^k$. Whether extra factors of $1/N$ appears or not depends on the form of $H_D + H_I(\lambda_b, [x_i])$, just as in our model extra factors of $1/N$ appeared through X -interaction. This was pointed out just below (88). We have to study more realistic detection process and fix the above correction terms, including the numerical estimates. In doing so, some simplification of the Hamiltonian will be required of course to make the problem tractable.

Our scheme can also be applied directly to the system described by the quantized field. Indeed, the field theory is much more suited to handle the macroscopic system, especially when taking the thermodynamic limit. Also, the measurement theory in the relativistic case can be studied using the field theory, since the formalism of the relativistic field theory is firmly established.

Finally, the most difficult problem of reduction, or einselection, is left out of the discussions in this paper. The extension of the dynamical reduction theory [15] to the relativistic case is controversial. However, in our case the application of the stationary phase to the relativistic field theory is straightforward. Also our results in this paper are independent of the precise mechanism of the reduction process, since we rely solely on the Schrödinger equation, which holds independently of how the reduction is realized.

APPENDIX A: NON-SEPARABLE CASE

To obtain the equation of motion for a macrovariable is the same problem of how to get the effective theory of a collective mode in many particle system. There are several methods but here we select the one which consists of inserting a delta-function in the path-integral. In the limit $N \rightarrow \infty$, it becomes equal to the method of the Legendre transformation [22]. We consider the case where X is given in collective notation $[x_i]$ as $g([x_i])/N$. Extension to the case where g includes $[p_i]$ is not difficult.

The signal function

Let us write the wave function of the detector plus object system at some fixed time as $\Psi([x_i], x)$. For any thermodynamically normal macroscopic system, the wave function has the form [21]

$$\Psi([x_i], x) = G e^{-F}, \quad (\text{A1})$$

where $F = F([x_i], x)$ is an extensive quantity and is of the order $O(N)$, so we write it as $F([x_i], x) = N\mathcal{F}([x_i], x)$. On the other hand, $G = G([x_i], x)$ describes the microscopic details. By normalizability, the real part of $\mathcal{F}([x_i], x)$ is positive definite and when $\mathcal{F}([x_i], x)$ changes by a finite amount, it describes macroscopically different state of the system. When we sum up wave functions having different values of \mathcal{F} , it represents the mixed state. The pure state is obtained by summing up various G 's, with the same \mathcal{F} . The signal function defined in (25) can be generalized to the non-separable case as follows.

$$\begin{aligned} J(X) &= N \int dx \int \prod_i dx_i \\ &\times |\Psi([x_i], x)|^2 \delta(NX - g([x_i])) \\ &= N \int dx \int \prod_i dx_i \int_{-\infty}^{\infty} dj \\ &\times \exp \left(-N2\text{Re}\mathcal{F}([x_i], x) + i\{NX - g([x_i])\}j \right) \\ &\equiv \int_{-\infty}^{\infty} dj \exp(-H(j) + iNjX) \equiv e^{-K(X)}. \quad (\text{A2}) \end{aligned}$$

We have dropped microscopic G term since it does not affect the stationary condition for X . Note here that $J(X)$ is real by definition, so $K(X)$ is a real quantity

and positive definite. Now $H(j)$ is extensive, so the function $H(j)$ and $K(X)$ are both extensive proportional to N . The results (A1) and (A3) have been obtained [21] by adopting the field theory to describe the macroscopic system, which automatically takes into account the statistical factor $1/N!$. The essential condition for these two equations to hold is the short range character of the interaction among constituent particles in the macroscopic system. Although the field theoretical approach is not taken in this paper, here we apply (A1) and (A3) to the quantum mechanical N -particle system, since these results express the general property of the thermodynamically normal system. Writing $K(X) = N\mathcal{K}(X)$, we first expand $\mathcal{K}(X)$ around the stationary solution $X = X^{\text{st}}$ satisfying $\partial\mathcal{K}(X)/\partial X = 0$. Suppose the second derivative at the stationary point is positive; $\mathcal{K}^{(2)}(X^{\text{st}}) > 0$. (Since $\mathcal{K}(X)$ is positive definite, at least one stationary point with $\mathcal{K}^{(2)} > 0$ exists. The solution with $\mathcal{K}^{(2)} < 0$ represents an unstable state.) In the limit $N \rightarrow \infty$, the analog of (3) for the case of positive definite $\mathcal{K}(X)$ holds;

$$\lim_{N \rightarrow \infty} e^{-N\mathcal{K}(X)} = e^{-N\mathcal{K}(X^{\text{st}})} \sqrt{\frac{2\pi}{-N\mathcal{K}''(X^{\text{st}})}} \delta(X - X^{\text{st}}).$$

This is equal to $\delta(X - X^{\text{st}})$ by normalization.

When $J(X)$ has the distribution in X , it represents the mixed state. Indeed, the signal function is rewritten as

$$\begin{aligned} J(X) &= \int dY J(Y) \delta(X - Y) \\ &= \lim_{N \rightarrow \infty} \int dY J(Y) \sqrt{N/\pi} \exp(-N(X - Y)^2). \end{aligned}$$

In this form, $J(X)$ of (12) is seen to be a superposition of different \mathcal{F} 's, with the weight $J(Y)$. In real detection process, the detector is so arranged that \mathcal{F} depends on the eigen-value λ_a of the object operator O to be measured. Writing the time t explicitly, $F(\lambda_a, [x_i], t) = N\mathcal{F}(\lambda_a, [x_i], t)$ and $K = N\mathcal{K}$ in (A3) becomes $N\mathcal{K}(\lambda_a, X, t)$. Then the stationary solution of $\partial\mathcal{K}/\partial X = 0$ depends on t and a ; $X = X_a^{\text{st}}(t)$. For the model of (18), this is written as $\xi_a(T)$ in (32).

The equation of motion of $X^{\text{st}}(t)$ itself can be obtained by the method of double paths Legendre transform, as discussed in detail in [22].

APPENDIX B: CALCULATION OF Φ_{ba}

By (28), (35) and (48), $J^{(1)}(X, T)$ is given by

$$\begin{aligned} J^{(1)}(X, T) &= \sum_b \Psi^{(0)*}(b, X, T) \Psi^{(1)}(b, X, T) + \text{c.c.} \\ &= \left(\frac{1}{\pi \Delta^2 \rho} \right)^{1/2} (-i/\hbar) \sum_{ab} C_b^*(H_O)_{ba} C_a \\ &\times \int_0^T ds \times \exp i\{(\theta_a - \theta_b)s\} \exp \Phi_{ba} + \text{c.c.} \quad (\text{B1}) \end{aligned}$$

By $X - \xi_b(T) = R_{ba}(X, T, 0)$, Φ_{ba} is expressed by

$$\begin{aligned} \Phi_{ba} = & -\frac{N^2 m^2}{2\hbar^2 T^2} \left(\frac{R_{ba}(X, T, s)^2}{D} + \frac{R_{ba}(X, T, 0)^2}{D^*} \right) \\ & + \frac{Nmi}{2\hbar T} \{R_{ba}(X, T, s)^2 - R_{ba}(X, T, 0)^2\} \\ & + \frac{i}{\hbar} N \{XQ_{ba}(T, s) + P_{ba}(T, s) \\ & - XQ_{ba}(T, 0) - P_{ba}(T, 0)\}. \end{aligned} \quad (B2)$$

Here we use (37) and rewrite the first term of Φ_{ba} as

$$\begin{aligned} & \frac{N^2 m^2}{2\hbar^2 T^2} \left(\frac{R_{ba}(X, T, s)^2}{D} + \frac{R_{ba}(X, T, 0)^2}{D^*} \right) \\ = & \frac{1}{\rho \Delta^2} \left(X - \frac{\xi_{ba}(T, s) + \xi_{ba}(T, 0)}{2} \right)^2 \\ & + \frac{1}{4\Delta^2 \rho} (\xi_{ba}(T, s) - \xi_{ba}(T, 0))^2 \\ & + \frac{iNm}{2\rho\hbar T} \{(X - \xi_{ba}(T, s))^2 - (X - \xi_{ba}(T, 0))^2\}. \end{aligned} \quad (B3)$$

Approximating $\rho = 1$, Φ_{ba} is now rewritten as

$$\begin{aligned} \Phi_{ba} = & -\frac{1}{\Delta^2} \left(X - \frac{\xi_{ba}(T, s) + \xi_{ba}(T, 0)}{2} \right)^2 \\ & - \frac{1}{4\Delta^2} (\xi_{ba}(T, s) - \xi_{ba}(T, 0))^2 + \frac{iN}{\hbar} \omega_{ba}(X, T, s), \end{aligned} \quad (B4)$$

with $\omega_{ba}(X, T, s)$ defined by

$$\begin{aligned} \omega_{ba}(X, T, s) = & X(Q(T, s) - Q(T, 0)) + P(T, s) - P(T, 0) \\ = & X(f_a - f_b)s - \frac{f_a^2}{6m} s^2 (3T - 2s) \\ & - \frac{f_b^2}{6m} \{(T - s)^3 - T^3\} - \frac{f_a f_b}{2m} s(T - s)^2. \end{aligned} \quad (B5)$$

APPENDIX C: CALCULATION OF $\Phi_{a'a;b}$

By (48), the explicit form of $\Phi_{a'a;b}$ defined in (64) is

$$\begin{aligned} \Phi_{a'a;b} = & -\frac{N^2 m^2}{2\hbar^2 T^2} \left(\frac{R_{ba}(T, s)^2}{D} + \frac{R_{ba'}(T, s')^2}{D^*} \right) \\ & + \frac{Nmi}{2\hbar T} \{R_{ba}(T, s)^2 - R_{ba'}(T, s')^2\} \\ & + \frac{i}{\hbar} N \{XQ_{ab}(T, s) + P_{ab}(T, s) \\ & - XQ_{a'b}(T, s') - P_{a'b}(T, s')\}. \end{aligned} \quad (C1)$$

As in (B3), we rewrite

$$\begin{aligned} & \frac{N^2 m^2}{2\hbar^2 T^2} \left(\frac{R(T, s)^2}{D} + \frac{R(T, s')^2}{D^*} \right) \\ = & \frac{1}{\rho \Delta^2} \left\{ \left(X - \frac{\xi_{ba}(T, s) + \xi_{ba'}(T, s')}{2} \right)^2 \right\} \\ & + \frac{1}{4\Delta^2 \rho} (\xi_{ba}(T, s) - \xi_{ba'}(T, s'))^2 \\ & - \frac{iNm}{2\rho\hbar T} \{(X - \xi_{ba}(T, s))^2 - (X - \xi_{ba'}(T, s'))^2\}. \end{aligned}$$

By setting $\rho = 1$, $\Phi_{a'a;b}$ is written as

$$\begin{aligned} \Phi_{a'a;b} = & -\frac{1}{\Delta^2} \left(X - \frac{\xi_{ba}(T, s) + \xi_{ba'}(T, s')}{2} \right)^2 \\ & - \frac{1}{4\Delta^2} (\xi_{ba}(T, s) - \xi_{ba'}(T, s'))^2 + \frac{iN}{\hbar} \omega_{a'a;b}(T, s), \end{aligned} \quad (C2)$$

$$\begin{aligned} \omega_{a'a;b}(T, s) = & \omega_{ba}(T, s) - \omega_{ba'}(T, s') \\ = & X(Q_{ba}(T, s) - Q_{ba'}(T, s')) + P_{ba}(T, s) - P_{ba'}(T, s'). \end{aligned} \quad (C3)$$

APPENDIX D: INTEGRATION BY X

Since each term in \sum_b of (61) is given by $K_b^{(1)} \delta(X - \xi_b(T))$, $K_b^{(1)}$ is calculable by integrating over X first and then by s for fixed b . After integration by X , the term

$$\exp \left[\frac{-1}{\Delta^2 \rho} \left\{ X - \frac{\xi_{ba}(T, s) + \xi_{ba}(T, 0)}{2} \right\}^2 + \frac{iN}{\hbar} X(f_a - f_b)s \right]$$

changes into

$$\begin{aligned} & \sqrt{\pi \Delta^2 \rho} \exp \left[-\frac{N^2 \Delta^2 \rho}{4\hbar^2} (f_a - f_b)^2 s^2 \right. \\ & \left. + \frac{Ni}{2\hbar} \left(\xi_{ba}(T, s) + \xi_{ba}(T, 0) \right) (f_a - f_b)s \right]. \end{aligned}$$

Then the dominant region of s -integration is $s = O(1/N)$. The second term in $[\dots]$ can be shown to be $O(s^3)$, if it is combined with $P_{ba}(T, s) - P_{ba}(T, 0)$. Indeed,

$$\begin{aligned} & (\xi_{ba}(T, s) + \xi_{ba}(T, 0))(f_a - f_b)s + P_{ba}(T, s) - P_{ba}(T, 0) \\ & \sim (f_a - f_b)(2f_a - f_b)s^3/6m. \end{aligned}$$

for small s . By $Ns = y$, s -integration now becomes

$$\begin{aligned} & \frac{1}{N} \int_0^{TN} dy \times \exp \{i(\theta_a - \theta_b)y/N\} \sqrt{\pi \Delta^2 \rho} \\ & \times \exp \left[-\frac{\Delta^2 \rho}{4\hbar^2} (f_a - f_b)^2 y^2 \right] + O(1/N^2). \end{aligned}$$

It gives $\hbar\pi/(N|f_a - f_b|)(1 + O(1/N))$. Thus we get

$$\int dX \Psi_b^{(0)*}(X, T) \Psi_b^{(1)}(X, T) = \frac{-i\sqrt{\pi} \sum_a C_b^*(H_O)_{ba} C_a}{N\Delta|f_a - f_b|}.$$

Adding term with c.c., we get (61) and (62).

APPENDIX E: NEGLECTED TERMS FOR $N \rightarrow \infty$

First, let us restrict the arguments up to $O([H_O]^{\text{nd}})^2$ and consider the correction to $|C_b|^2$, A_b and B_b of (85). 1. $|C_b|^2$: As stated in Sec.III E, the correction comes from the diffusion. Including the diffusion, Δ is replaced by $\sqrt{\rho}\Delta$. Since $\sqrt{\rho} = 1 + O(1/N^2)$, $|C_b|^2$ changes into $|C_b|^2(1 + O(1/N^2))$. Thus the neglected terms are $O(1/N^2)$. The fact that it is small numerically has been

checked in Sec.III E.

2. A_b ; The correction comes from the higher order of the fluctuation around the stationary path, which is down by $O(1/\sqrt{N})$ compared with the term retained. In obtaining A_{b1} of (86), both s and X have fluctuations of $O(1/\sqrt{N})$. As for A_{2b} , s and s' has the same size of the fluctuations but X has no stationary point. Therefore $A_{b1,2}$ are replaced by $A_{b1,2}(1 + O(1/\sqrt{N}))$. The leading order of neglected terms in A_b/N is thus $O(1/N\sqrt{N})$.

3. B_b ; Besides s and s' having the fluctuation of $O(1/\sqrt{N})$, X fluctuates near the stationary phase with the size of $O(1/N^{1/3})$, see (79). Thus B_b is replaced by

$B_b(1 + O(1/N^{1/3}))$ and the corrections to $B_b/NN^{1/3}$ is $O(1/NN^{1/3})(1/N^{1/3})$.

Next, consider the term $([H_O]^{\text{nd}})^k$ for $k \geq 3$. In the expansion (14), one factor of $[H_O]^{\text{nd}}$ accompanies an integration over the parameter s . Integration around the stationary phase produces one factor $1/\sqrt{N}$, so $(1/\sqrt{N})^k$ appears. Extra factor of some power of $1/N$ may appear depending on the result of the X -integration, as stated above. Therefore, we can say that $([H_O]^{\text{nd}})^k$ term has the power of at least $(1/\sqrt{N})^k$.

Summarizeibg, eq.(85) is correct up to the order retained there, the leading correction being $O(1/N\sqrt{N})$.

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